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SENSITIVITY ANALYSIS OF MECHANICAL SYSTEMS WITH UNILATERAL CONSTRAINTS†

B. R. KLEPFISH
Rostov-on-Don
e-mail: klepfish@mail.don.sitetc.net

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The problem of sensitivity analyses for mechanical systems with unilateral constraints is considered. The problem is formulated as one of computing the derivative, with respect to the vector of parameters, of a functional characterizing the motion. To compute the derivative, the adjoint variable approach is extended to systems with unilateral constraints. The set of active constraints may change in the course of the motion, with or without impact. Jump conditions for the adjoint variables are indicated for the times at which changes occur in the set of active constraints. As an example, a mechanical system whose motion is constrained by an absolutely elastic stop is considered.

Sensitivity analysis is used in problems which arise when it is desired to assess the influence of measurement errors or inaccuracies in estimates of external forces acting on real systems [1–3]. There are also applications of sensitivity analysis to projection problems, modelled either by systems of ordinary differential equations or by partial differential equations [1, 2]. However, these applications are concerned mainly with problems without constraints or with bilateral constraints. At the same time, there is a large class of systems with unilateral constraints for which it seems interesting to use sensitivity analysis. The monographs [4, 5], which are devoted to systems with unilateral constraints, present examples of vibro-elastic systems, systems of cyclic automation [4], and systems that arise when solving problems in the dynamics of structures [5]. The most common sensitivity measure is the derivative with respect to a parameter of some functional that can be computed along the trajectories of motion of the system [3, 6]. General approaches to the computation of such derivatives along trajectories of dynamical systems are known [7, 8]. In this paper formulae are derived for the derivative of an integral functional which characterizes the sensitivity of a mechanical system with unilateral constraints in the case of the impact of the system against one constraint or release from the constraint.

1. EQUATIONS OF MOTION

Consider a system of ordinary differential equations describing the motion of a mechanical system with ideal unilateral holonomic constraints, confining our attention to finite inequality constraints

$$M\ddot{q} = F(q, \dot{q}, t) + G\lambda, \quad \Phi(q) \geq 0, \quad G = \partial\Phi(q)/\partial q \in R^{n\times m}, \quad t \in [t_1, T] \tag{1.1}$$

where $q \in R^n$ is the vector of generalized coordinates, $M \in R^{n\times n}$ is the symmetric positive-definite matrix of the masses, $F(q, \dot{q}, t)$ is the vector of forces, $\Phi: R^n \to R_+^m$ is a continuously differentiable vector function whose components describe holonomic stationary unilateral constraints and $\lambda(t)$ is the vector of Lagrange multipliers.

The Lagrange multiplier λ_i corresponding to the ith constraint satisfies the following complementarity condition [9, 10].

$$\lambda_i \geq 0, \quad \Phi_i \geq 0, \quad \lambda_i\Phi_i = 0 \tag{1.2}$$

We shall consider system (1.1) in the interval $[t_1, T]$ as a composite or a system with variable structure. To that end, the whole interval is divided into subintervals

$$[t_j, t_{j+1}], \quad j = 1, 2, \ldots, N-1, \quad t_N = T \tag{1.3}$$

within each of which the set of active constraints does not change. The union of these subintervals is the entire time interval $[t_1, T]$.

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To compute the trajectories of motion of the mechanical system, the correspondence between the sets of active constraints and the time subintervals (1.3) must be determined, and the conditions governing the relation between the values of the generalized velocities at the end-points of the subintervals (transition with or without impact) must be determined. To determine the equations of motion of system (1.1), that is, to indicate the set of active constraints in each subinterval, one generally uses a generalization of Gauss's principle of least constraint [11] to systems with unilateral constraints [12], or considers the mechanical system as a system satisfying a complementarity condition [9, 10].

Gauss's principle of least constraint leads to the optimization problem

$$\min_{\ddot{q}} \frac{1}{2}(\ddot{q} - M^{-1}F)^T M(\ddot{q} - M^{-1}F) \tag{1.4}$$

under constraints

$$G^T\ddot{q} + \dot{G}^T\dot{q} \geq 0 \tag{1.5}$$

The solution of this quadratic programming problem with linear constraints satisfies the Kuhn–Tucker Theorem [13], whose geometrical meaning is that, for an optimal solution, the antigradient of a suitable minimization of the expression (1.4) may be expressed as a non-negative linear combination ($\lambda_i \geq 0$) with the sign of the columns of the matrix G corresponding to active constraints reversed. Consequently, the matrix G corresponding to the interval (1.3) may be obtained from the Kuhn–Tucker conditions. The Kuhn–Tucker system for a quadratic programming problem may be reduced to a linear complementarity problem. Consequently, problem (1.4), (1.5) may be solved by using the procedures of [12, 13]. Complementarity problems are widely used to describe mechanical systems with unilateral constraints [10, 12, 14].

After determining the set of columns of the matrix G corresponding to a subinterval (1.3), it is convenient for the integration of Eqs (1.1) to exclude the multiplier λ [15].

We will now consider the relation between the values of the generalized velocities of the mechanical system at the common end-point of two adjacent subintervals. The transition from one subinterval of the time of motion to another may be of either of two types: with or without impact. To obtaining defining relations in impact theory one most frequently makes use of Newton's hypothesis [4, 16]: the change in the normal component of the generalized velocity of colliding bodies depends only on their materials, not on the velocities, and satisfies the relation [4, 16]

$$g^T\dot{q}(t^+) = -eg^T\dot{q}(t^-) \tag{1.6}$$

where e is the coefficient of restitution of the velocity on impact, which varies from zero to unity, and g is the gradient of the equation of the constraint.

The change in velocities on impact at time t_k against a constraint satisfying condition (1.6) may be expressed in the form

$$\dot{q}(t_k^+) = \dot{q}(t_k^-) - (1+e)\frac{g^T\dot{q}(t_k^-)}{g^T M^{-1} g} M^{-1} g \tag{1.7}$$

In the case of absolutely elastic impact ($e = 1$) this formula, for scleronomous systems with one unilateral constraint, is identical with formula (13.15) in [4].

Thus, the system considered in this paper may be classified as a mechanical system with unilateral constraints – inequality constraints on the generalized coordinates. At the impact times, the generalized coordinates of the system are continuous, but the generalized velocities experience jumps. When the generalized velocities change as a result of impact, their initial values for the appropriate subinterval (1.3) are evaluated using formula (1.7), depending on the assumptions adopted concerning the value of the coefficient $e = 1$. Within each subinterval (1.3) the system satisfies equations of motion of the form (1.1).

2. SENSITIVITY ANALYSIS

Sensitivity analysis will be carried out below for the case in which the mechanical system impacts one constraint and the case in which it is released from the constraint. We shall assume that the equations of motion of the system have been reduced to a first-order system of ordinary differential equations and, for simplicity, that the time interval contains two instants at which the set of active constraints changes. We will write the equations of the motion in the form

$$dx/dt = f_i(x, b), \quad t_i \leq t \leq t_{i+1}, \quad i = 1, 2, 3; \quad x(t_1) = x_1, \quad t_4 = T \tag{2.1}$$

where t_2 and t_3 are the times at which the set of active constraints changes, once without impact and once with impact, respectively; $b = (b_1, \ldots, b_N)^T$ is the parameter vector. For this system obviously $x^T = (q, \dot{q})^T$.

We will take as the measure of the sensitivity the value of the derivative of the integral functional

$$\Omega = \int_{t_1}^{T} P(x(t), b)dt \tag{2.2}$$

with respect to the parameter b [2, 3, 6], which is evaluated using an approach developed in optimal control theory [7, 8]. Formulae for the sensitivity analysis of systems with bilateral constraints were derived using this approach, e.g. in [6].

Suppose the value of the parameter vector b^0 is perturbed by a small quantity δb, where $\|\delta b\|$ is a small quantity, and the trajectory $x(t)$ is such that the set of active constraints changes at unspecified times t_2 and t_3. The times are determined by the conditions for the release from one of the constraints (t_2) and impact against the other (t_3). Then the equation in variations for δx will be

$$\frac{d}{dt}\delta x - \frac{\partial f_i}{\partial x}\delta x = \frac{\partial f_i}{\partial b}\delta b, \quad t \in [t_i^+, t_{i+1}^-], \quad i = 1, 2, 3 \tag{2.3}$$

where $t_{i+1}^-(t_i^+)$ is the left-hand (right-hand) limit of the time at which the set of active constraints changes.

The adjoint variables $\psi(t)$ will be given specified [7] as a solution of the boundary-value problem

$$\frac{d\psi}{dt} + \frac{\partial f_i^T}{\partial x}\psi = -P_x, \quad \psi(T) = 0 \tag{2.4}$$

where f_i corresponds to the time subinterval and the jump conditions [17] for $\psi[t]$ at t_2 and t_3 will be given below.

Lagrange's identity [7], used in deriving the formulae within the framework of the approach adopted here, will be written as

$$\sum_1^3 \int_{t_i^+}^{t_{i+1}^-} \left(\psi\left(\frac{d\delta x}{dt} - \frac{\partial f_i}{\partial x}\delta x\right) + \delta x\left(\frac{d\psi}{dt} + \frac{\partial f_i^T}{\partial x}\psi\right)\right)dt = \psi\delta x|_{t_1}^{T} - \psi\delta x|_{t_2^-}^{t_2^+} - \psi\delta x|_{t_3^-}^{t_3^+} \tag{2.5}$$

Since the vector $x(t_1)$ is specified and $x(T)$ is free, it follows from the transversality conditions for functionals of type (2.2) [7] that $\psi\delta x|_{t_1}^{T} = 0$. The identity (2.5), together with Eq (2.3), is used so that, in the expression for the derivative of the functional (2.2) with respect to the parameter b, terms containing the factor δx will be expressed in terms of the corresponding terms with the factor δb. Namely, according to (2.3) and (2.4), equality (2.5) may be rewritten as

$$\sum_1^3 \int_{t_i^+}^{t_{i+1}^-} \psi\frac{\delta f_i}{\delta b}dt\delta b + \psi\delta x|_{t_2^-}^{t_2^+} + \psi\delta x|_{t_3^-}^{t_3^+} = \int_{t_1}^{T} P_x(t)\delta x(t)dt \tag{2.6}$$

We now obtain jump conditions [17] for the adjoint variables at times t_2 and t_3. Let δt_2 be the variation of the time t_2 corresponding to variation of the parameter δb. It then follows from the condition of continuity of the coordinates and velocities at the instant of release from the constraint that the following relations hold for the variation of the coordinates [7, 8]

$$\delta x(t_2^+) = \delta x(t_2^-) + (f_1 - f_2)\delta t_2 \tag{2.7}$$

As remarked above, the subscripts 1 and 2 correspond to the subintervals of time in which the motion of the mechanical system is taking place. The released constraint is denoted by $\Phi_j(q) = R(x)$. It is obvious that, at least in a small neighbourhood of t_2, we have the inequality $d\Phi_j/dt = R_x f_i > 0$, where the subscript i corresponds to the subinterval of the time of motion.

Using the first terms of the series expansion of the relation $R(x(t_2^- + \delta t_2)) = 0$, we obtain [7] a relation between δt_2 and $\delta x(t_2^-)$. Substitution of this relation and formula (2.7) into Eq. (2.6) shows that the quantity

$$\psi \delta x \Big|_{t_2^-}^{t_2^+} \tag{2.8}$$

vanishes to within the accuracy necessary for computing the derivative, provided [7] that

$$\psi(t_2^-) = \psi(t_2^+) - \frac{(f_1 - f_2, \psi(t_2^+))}{R_x f_1} R_x \tag{2.9}$$

The specific notation (2.9) of the jump conditions [17] for $\psi(t_2^+)$ is particularly convenient, since it explicitly defines the passage from $\psi(t_2^+)$ to $\psi(t_2^-)$ in integration from right to left [7].

We will now derive a jump condition for the adjoint variables at time t_3. Suppose there is an impact at time t_3, and accordingly a discontinuity in the values of the velocities $\dot{q}(t_3^{\mp})$. The new values of the velocities at time t_3^+ are determined by formulae of type (1.7). Combining formulae of type (1.7) for the generalized velocities with the continuity conditions for the generalized coordinates at time t_3, we obtain a relation, an interior-point condition [8], which we write as

$$N(x(t_3^-), x(t_3^+)) = 0 \tag{2.10}$$

In the case of discontinuous phase coordinates, the variation of a functional of type (2.2) contains a term of the form $(P(x(t_3^-), b^0) - P(x(t_3^+), b^0))\delta t_3$; it reflects the variation of the t_3. As a result of using Lagrange's identity, in the variation of a functional of type (2.2) both terms reflecting the variation of t_3 and terms of type (2.8) for $t_3^{\mp}$ occur. To guarantee that the sum

$$\psi \delta x \Big|_{t_3^-}^{t_3^+} + (P(x(t_3^-), b^0) - P(x(t_3^+), b^0))\delta t_3 = 0 \tag{2.11}$$

will vanish, we use an expression representing the condition for arrival at the constraint at which the impact occurs,

$$\Phi_j(q(t_3)) = R_1(x(t_3)) = 0 \tag{2.12}$$

as well as conditions (2.10). To do this, we expand expressions (2.10) and (2.12)

$$\begin{aligned} R_1(x(t_3^+ + \delta t_3)) &= R_1(x(t_3^+)) + R_{1x} dx(t_3^+) \\ N(x(t_3^- + \delta t_3), x(t_3^+ + \delta t_3)) &= N(x(t_3^-), x(t_3^+)) + \frac{\partial N}{\partial x(t_3^-)} dx(t_3^-) + \frac{\partial N}{\partial x(t_3^+)} dx(t_3^+) \end{aligned} \tag{2.13}$$

and add to these equations the relation between the variations of the coordinates and the time t_3

$$dx(t_3^{\pm}) = \delta x(t_3^{\pm}) + \dot{x}(t_3^{\pm})\delta t_3. \tag{2.14}$$

where

$$\dot{x}(t_3^-) = f_2, \quad \dot{x}(t_3^+) = f_3$$

Using relations (2.14) and (2.13), we can express δt_3 and $\delta x(t_3^-)$ in terms of $\delta x(t_3^+)$. Substituting the expressions obtained into Eq. (2.11), we conclude that the latter will surely be valid if the vector $\psi(t_3^-)$ satisfies the following system of $2n$ equations

$$\begin{aligned} &\left[\left(\frac{\partial N}{\partial x(t_3^-)}\right)^{-1}\left(\left(\frac{\partial N}{\partial x(t_3^-)}\dot{x}(t_3^-) + \frac{\partial N}{\partial x(t_3^+)}\dot{x}(t_3^+)\right)\right)\frac{R_{1x}^T}{R_{1x}\dot{x}(t_3^+)} - \frac{\partial N}{\partial x(t_3^+)}\right]^T \psi(t_3^-) = \\ &= \frac{P(x(t_3^-), b^0) - P(x(t_3^+), b^0)}{R_{1x}\dot{x}(t_3^+)} R_{1x} - \psi(t_3^+) \end{aligned} \tag{2.15}$$

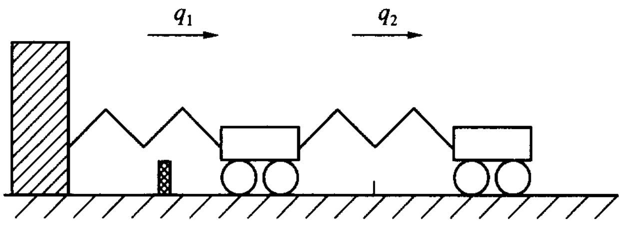

Fig. 1

The form of system (2.15), like that of (2.9), is convenient for integrating system (2.4) from right to left.

Finally, the derivative of the functional with respect to the parameter b has the form

$$\frac{\partial\Omega}{\partial b} = \int_{t_1}^{T} P_b(t)dt + \Sigma \int_{t_i^+}^{t_{i+1}^-} \psi\frac{\partial f_i}{\partial b}dt \tag{2.16}$$

where the variables $\psi(t)$ satisfy the equation and boundary condition (2.4), as well as the jump conditions (2.9) and (2.15).

Thus, the problem of computing the derivative of the functional (2.2) for a mechanical system with a variable set of active constraints is considered successively in subintervals with varied limits of integration. At the end-points of the subintervals, the adjoint variables $\psi(t)$ satisfy jump conditions [17]. The case of a larger number of subintervals is considered in a similar way.

3. EXAMPLE

To illustrate the approach just proposed, we present a sensitivity analysis for a mechanical system [14] consisting of two carts connected by a spring (see Fig. 1). The left cart is also attached to a wall by a spring. In addition, the motion of the left cart is limited by an absolutely elastic stop, whose position corresponds to the undeformed state of the spring attaching that cart to the wall.

Let q_j be the deviations of the carts from the positions characterized by the undeformed state of the springs, m_j be the masses of the carts, k_j be the stiffnesses of the springs (the subscript $j = 1$ corresponds to the left cart and $j = 2$ to the right one), and λ be the Lagrange multiplier corresponding to the unilateral constraint arising from the contact between the left cart and the stop. Then the equations of motion are

$$m_1\ddot{q}_1 = -k_1q_1 - k_2(q_1 - q_2) + \lambda, \quad m_2\ddot{q}_2 = -k_2(q_2 - q_1)$$

The unilateral constraint corresponding to the stop situated at the point $q_1 = 0$ guarantees that the coordinate q_1 will be non-negative. Consequently, the reaction of the constraint is non-negative, that is, $\lambda \geq 0$. If $q_1 > 0$, $\lambda = 0$, if the left cart is not touching the stop, the reaction force of the constraint is zero. At times when $q_1 = 0$ the system will experience the action of additional instantaneous forces. The equations of motion of the system are as follows:

free motion

$$\begin{gathered} m_1\ddot{q}_1 = -k_1q_1 - k_2(q_1 - q_2), \quad m_2\ddot{q}_2 = -k_2(q_2 - q_1) \\ t_1 \leq t \leq \tau, \quad \tau + \Delta t \leq t \leq T \end{gathered} \tag{3.1}$$

motion under the action of the constraint

$$\begin{gathered} m_1\ddot{q}_1 = -k_1q_1 - k_2(q_1 - q_2) + \lambda, \quad m_2\ddot{q}_2 = -k_2(q_2 - q_1) \\ \tau \leq t < \tau + \Delta t \end{gathered} \tag{3.2}$$

where τ is the time at which the left cart makes contact with the stop. The variation in the values of the velocities must be evaluated by formula (1.7) with $e = 1$.

As the functional whose sensitivity is to be analysed we take

$$\Omega = \int_{t_1}^{T} (\dot{q}_1^2 + \dot{q}_2^2) dt \tag{3.3}$$

This quantity characterizes the velocity of the system in the interval of motion. We shall analyse the sensitivity of the functional (3.3) to the constants k_1 and k_2 of the springs. Since the integrand in (3.3) does not depend on these parameters, the derivative of the functional Ω has the form

$$\frac{\partial \Omega}{\partial (k_1, k_2)} = \int_{t_1}^{\tau^-} \psi^T \frac{\partial f_1}{\partial (k_1, k_2)} dt + \int_{\tau^+}^{T} \psi^T \frac{\partial f_2}{\partial (k_1, k_2)} dt \tag{3.4}$$

The variables f_1 and f_2 correspond to the equations of free motion in the intervals $[t_1, \tau)$ and $[\tau, T]$, respectively, and the vector ψ is a solution of the adjoint system (2.4). At time τ the velocity vector and the vector of adjoint variables experience a discontinuity. Recall the formula for the re-evaluation of the velocities has the form (1.7), $e = 1$. After obvious reduction, we obtain

$$\dot{q}^+(\tau) = (-\dot{q}_1^-, \dot{q}_2^-)^T$$

Thus, after impact against the absolutely elastic obstacle at the point $q_1 = 0$, the velocity of the left cart changes sign, but the velocity of the right cart remains unchanged. To evaluate the values of the adjoint variables that satisfy the jump conditions, one has to use formula (2.15).

The sensitivity analysis was carried out for parameter values $m_1 = m_2 = 1$, the system being considered in the time interval [0, 2]. The initial data were taken to be

$$q_1(0) = 1, \quad \dot{q}_1(0) = 1, \quad q_2(0) = 2, \quad \dot{q}_2(0) = 0$$

The quantities k_1 and k_2 were chosen on a section of the straight line

$$(2.5, 2.5)^T + \alpha(1, 1)^T, \quad \alpha \in [0, 0.4]$$

with stepsize 0.1 in α. The results of the computations are shown below.

$k_1 = k_2$	2.5	2.6	2.7	2.8	2.9
Ω	5.962	6.213	6.438	6.639	6.819
$\partial\Omega/\partial k_1$	1.585	1.687	1.790	1.888	1.974
$\partial\Omega/\partial k_2$	0.926	0.919	0.912	0.905	0.897
$-\Delta\Omega \times 10^3$	–	0.1	36	69	102
$-(\Delta\Omega/\Omega) \times 10^3$	–	0.01	6	10	15

It can be seen that positivity of the sensitivity coefficients corresponds to an increase in the values of Ω. The first sensitivity coefficient is greater than the second for all values of k_1 and k_2, since $\partial\Omega/\partial k_1$ corresponds to the left cart, which collides with the stop. At the same time, as the values of k_1 and k_2 increase, so do the values of $\partial\Omega/\partial k_1$, but those of $\partial\Omega/fk_2$ decrease. We also list in the table scaled values of the errors in the computed quantities $\Delta\Omega \times 10^3$, that is, the differences between the computed values of Ω and the values obtained by linear approximation using the sensitivity coefficients, namely.

$$\Delta\Omega = \Omega\left(\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} + \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}\right) - \Omega\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} - \left(\frac{\partial \Omega}{\partial (k_1, k_2)}\right)^T \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$$

In the last row we give the scaled relative errors, that is, the quantities $(\Delta\Omega/\Omega) \times 10^3$, which may be regarded as relative error estimates for the method. Since their values do not exceed 2%, the evaluation of the method is positive.

Thus, the approach proposed here enables as to estimate the directions of increase (or decrease) of an index characterizing the motion of the system in the time interval when the parameter values are changed. In addition, the values of the sensitivity coefficients indicate the relative contribution of changes in the values of the parameters to the change in the value of (3.3). This approach can be utilized to analyse certain aspects of the motion of systems of the cyclic automation and vibro-elastic systems. The enables one to obtain well-founded estimates for the motion of such systems taking collisions into account, and also to estimate the influence of measurement errors in the motion of these systems, that is, to make fuller use of the reserves of kinematic schemes according to a selected performance index.

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